Traveling fronts for lattice neural field equations

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Excitatory neural network on a lattice

Lattice Neural Field Equation (LNFE)

$$\dot{u}_n(t) = -u_n(t) + \sum_{j \in \mathbb{Z}} K_j S(u_{n-j}(t)), \quad (n,t) \in \mathbb{Z} \times (0,\infty)$$

- $u_n(t) \in \mathbb{R}$: membrane potential of neuron labelled n at time t;
- $K_j \geq 0$: strength of interactions (all to all – infinite range)
- $u \mapsto S(u)$: firing rate function (of sigmoidal type)
- discrete version of

$$\partial_t \mathbf{u}(x,t) = -\mathbf{u}(x,t) + \int_{\mathbb{R}} \mathcal{K}(y) S(\mathbf{u}(x-y,t))\mathrm{d}y$$

References: Wilson-Cowan '72, Amari '77, Hopfield'84, Ermentrout '98, Coombes '05, Bressloff '12

Traveling waves

Traveling wave: speed $c \in \mathbb{R}$

$$u_n(t) = \mathbf{u}(n - ct), \quad \mathbf{u} : \mathbb{R} \to \mathbb{R}$$

Traveling wave equation: $x = n - ct$

$$-c\mathbf{u}'(x) = -\mathbf{u}(x) + \sum_{j \in \mathbb{Z}} K_j S(\mathbf{u}(x - j)), \quad x \in \mathbb{R}, \tag{1a}$$

$$\lim_{x \to -\infty} \mathbf{u}(x) = 1 \text{ and } \lim_{x \to +\infty} \mathbf{u}(x) = 0, \tag{1b}$$

- FDE of mixed type (ill posed as a Cauchy problem)
- dynamics depend on (all) past $j \leq 0$ and (all) future $0 \leq j$

It is assumed that when $c = 0$ equation (1) is an infinite recurrence.

Assumptions

On the firing rate function S:

Hypothesis (H1) - Bistable nonlinearity. *We suppose that:*

(i) $S \in \mathscr{C}_b^r(\mathbb{R})$ *for* $r \geq 2$ *with* $S(0) = 0$ *and* $S(1) = 1$ *together with* $S'(0) < 1$ *and* $S'(1) < 1$*;*

(ii) *there exists a unique* $\theta \in (0,1)$ *such that* $S(\theta) = \theta$ *with* $S'(\theta) > 1$*;*

(iii) $u \mapsto S(u)$ *is strictly nondecreasing on* $[0,1]$ *and there exists* $s_m > 1 > s_0 > 0$ *such that* $s_0 < S'(u) \leq s_m$ *for all* $u \in [0,1]$.

On the weights $(K_j)_{j\in\mathbb{Z}}$:

Hypothesis (H2) - Weights. *We suppose that:*

(i) $\sum_{n\in\mathbb{Z}} K_n = 1$ *;*

(ii) *for all* $n \in \mathbb{Z}$*, we have* $K_n = K_{-n} \geq 0$ *and* $K_{\pm 1} > 0$*;*

(iii) $\sum_{n\in\mathbb{Z}} |n| K_n < \infty$.

Existence of monotone traveling waves

Theorem

Suppose that the Hypotheses (H1)-(H2) are satisfied then there exists a traveling wave solution $u_n(t) = \mathbf{u}_*(n - c_* t)$ of LNFE such that the profile $\mathbf{u}_*$ satisfies the traveling wave problem (1)*.*
Moreover:

(i) $\mathrm{sgn}(c_*) = \mathrm{sgn} \int_0^1 (-u + S(u))\mathrm{d}u$ *if* $c_* \neq 0$;

(ii) *if* $\int_0^1 (-u + S(u))\mathrm{d}u = 0$ *then* $c_* = 0$;

(iii) *if* $c_* \neq 0$ *then* $\mathbf{u}_* \in \mathscr{C}^{r+1}(\mathbb{R})$ *and* $\mathbf{u}_*' < 0$ *on* $\mathbb{R}$;

(iv) *if* $c_* = 0$ *we denote* $(\tilde{\mathbf{u}}_n^*)_{n\in\mathbb{Z}}$ *the stationary wave solution, then* $(\tilde{\mathbf{u}}_n^*)_{n\in\mathbb{Z}}$ *is a strictly decreasing sequence.*

Sketch of the proof

The key idea (Bates & Chen '99) is to regularize the traveling wave equation

$$-c\mathbf{u}'(x) = -\mathbf{u}(x) + \sum_{j\in\mathbb{Z}} K_j S(\mathbf{u}(x-j))$$

Let $\Psi \in \mathscr{C}^\infty(\mathbb{R})$, $\Psi \geq 0$, $\int_{\mathbb{R}} \Psi(x)\mathrm{d}x = 1$, even and with compact support

$$\rho_m(x) := m\Psi(mx), \quad \mathcal{K}_m(x) := \sum_{j=-m}^{m} \frac{1}{\omega_m} K_j \rho_m(x-j), \text{ with } \omega_m := \sum_{j=-m}^{m} K_j$$

New traveling wave problem

$$-c_m\mathbf{u}_m' = -\mathbf{u}_m + \mathcal{K}_m * S(\mathbf{u}_m), \text{ on } \mathbb{R}, \tag{2a}$$

$$\lim_{x\to-\infty} \mathbf{u}_m(x) = 1 \text{ and } \lim_{x\to+\infty} \mathbf{u}_m(x) = 0. \tag{2b}$$

Apply the results of Ermentrout & McLeod '93 to get a monotone solution $(\mathbf{u}_m, c_m)$ of (2) then pass to the limit $m \to +\infty$.

Uniqueness of traveling waves with nonzero speed

Theorem

Let $(\mathbf{u}_*, c_*)$ be a solution to (1) as given in Theorem 1, such that $c_* \neq 0$. Let $(\hat{\mathbf{u}}, \hat{c})$ be another solution to (1). Then $c = \hat{c}$ and, up to a translation, $\mathbf{u}_* = \hat{\mathbf{u}}$.

Ideas of the proof:

- Construct appropriate sub and super solutions of the form

$$w_n^{\pm}(t) := \mathbf{u}_* \left(n - c_* t + \xi_0 \mp \sigma\gamma(1 - e^{-\beta t}) \right) \pm \gamma e^{-\beta t}, \quad \forall n \in \mathbb{Z}$$

  for some parameters ξ_0, σ, γ and $2\beta = \min\{1 - S'(0); 1 - S'(1)\}$.
- Use comparison principle and a "squeezing" technique to prove uniqueness
- **Reference:** Chen'97

Spectral stability

LNFE in moving coordinate:

$$-c\mathbf{u}'(x) = -\mathbf{u}(x) + \sum_{j\in\mathbb{Z}} K_j S(\mathbf{u}(x-j))$$

Linearized operator around $(\mathbf{u}_*, c_*)$ with $c_* \neq 0$:

$$\mathcal{L} : H^1(\mathbb{R}) \to L^2(\mathbb{R}), \qquad \mathcal{L}\mathbf{v} := c_*\mathbf{v}' - \mathbf{v} + \mathcal{K}_\delta * [S'(\mathbf{u}_*)\mathbf{v}],$$

where $\mathcal{K}_\delta * \mathbf{w} = \sum_{j\in\mathbb{Z}} K_j \mathbf{w}(\cdot - j)$.

Floquet-like spectral structure:

$$\mathcal{L}(e^{2\pi \mathbf{i} x}\mathbf{u}) = e^{2\pi \mathbf{i} x}(2\pi \mathbf{i} c_* + \mathcal{L})\mathbf{u}$$

- spectrum invariant under shifts by $2\pi \mathbf{i} c_*$
- lattice doesn't feel oscillations on scale smaller than distance in lattice

Spectral properties of $\mathcal{L}$

Hypothesis (H2η) - Exponential localization. *We suppose that:*

(i) $(K_j)_{j\in\mathbb{Z}}$ *satisfies (H2);*

(ii) *there exists* $\eta > 0$, *such that* $\sum_{j\in\mathbb{Z}} K_j e^{\eta|j|} < \infty$.

Proposition

Assume that Hypotheses (H1)-(H2η) and that $(\mathbf{u}_*, c_*)$ *is the traveling wave solution given in Theorem 1 with* c_*. *We have:*

- 0 *is an algebraically simple eigenvalue of* $\mathcal{L}$ *with a negative eigenfunction* $\mathbf{u}_*'$;
- *the adjoint operator* $\mathcal{L}^*$ *has a negative eigenfunction, denoted* $\mathbf{q} \in \mathscr{C}^1(\mathbb{R})$, *corresponding to the simple eigenvalue* 0;
- *for all* $0 < \kappa < 2\beta$ *the operator* $\mathcal{L} - \lambda$ *is invertible as an operator from* $H^1(\mathbb{R})$ *to* $L^2(\mathbb{R})$ *for all* $\lambda \in \mathbb{C}\backslash 2\pi\mathbf{i}c_*\mathbb{Z}$ *such that* $\Re(\lambda) \geq -\kappa$;
- *there exist* $\eta_*, \eta_{**} \in (0, \eta)$ *and some constants* $C_* > 0$, $C_{**} > 0$ *such that*

$$|\mathbf{u}_*'(x)| \leq C_* e^{-\eta_*|x|} \|\mathbf{u}_*\|_{L^\infty(\mathbb{R})}, \text{ and } |\mathbf{q}(x)| \leq C_{**} e^{-\eta_{**}|x|} \|\mathbf{q}\|_{L^\infty(\mathbb{R})}.$$

Toward nonlinear stability

GOAL: given existence and spectral stability, prove nonlinear stability

LNFE can be written

$$\dot{\mathfrak{u}}(t) = \mathcal{F}(\mathfrak{u}(t)), \text{ with } \mathfrak{u} = (u_n)_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$$

where $\bar{\mathfrak{u}}(t) = (\mathbf{u}_*(n - c_* t))_{n\in\mathbb{Z}}$ with $c_* \neq 0$ is a solution with

$$\bar{\mathfrak{u}}_n(t) = \bar{\mathfrak{u}}_{n-1}\left(t - \frac{1}{c_*}\right), \quad n \in \mathbb{Z}$$

- TW is relative periodic orbit
- Linearization:
$$\dot{\mathfrak{v}}(t) = D\mathcal{F}(\bar{\mathfrak{u}}(t))\mathfrak{v}(t)$$

$\Rightarrow$ use spectral information on $\mathcal{L}$ to obtain decay estimates on the flow of $\dot{\mathfrak{v}}(t) = D\mathcal{F}(\bar{\mathfrak{u}}(t))\mathfrak{v}(t)$ and prove nonlinear stability

Main strategy

1. Moving coordinate frame:

$$\partial_t \mathbf{v} = \mathcal{L}\mathbf{v}, \quad (\mathcal{L} - \lambda)\mathbf{G} = \delta(\cdot - x_0),$$

   - spectral properties on $\mathcal{L}$
   - resolvent kernel $\mathbf{G}_\lambda(x, x_0)$

2. Coordinates of the lattice:

$$\dot{\mathfrak{v}}(t) = D\mathcal{F}(\bar{\mathfrak{u}}(t))\mathfrak{v}(t), \quad \mathfrak{v}(t_0) = (\delta(j - j_0))_{j\in\mathbb{Z}} \quad \Rightarrow \text{ solution } \mathfrak{v}^{t_0 j_0}(t)$$

   - Green's function

$$\mathcal{G}_{jj_0}(t, t_0) = \mathfrak{v}_j^{t_0 j_0}(t)$$

   - Relationship (Benzoni-Gavage, Huot, Rousset '03)

$$\mathcal{G}_{jj_0}(t, t_0) = \frac{1}{2\pi\mathbf{i}} \int_{R-\mathbf{i}\pi c_*}^{R+\mathbf{i}\pi c_*} e^{\lambda(t-t_0)}\mathbf{G}_\lambda(j - c_* t, j_0 - c_* t_0)\mathrm{d}\lambda \quad R \gg 1$$

Green's function representation

Proposition

Assume that Hypotheses (H1)-(H2η) and that $(\mathbf{u}_*, c_*)$ is the traveling wave solution given in Theorem 1 with c_*. Then there exists $\epsilon > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \epsilon$, we have the representation

$$\mathbf{G}_\lambda(x, x_0) = \mathbf{E}_\lambda(x, x_0) + \widetilde{\mathbf{G}}_\lambda(x, x_0),$$

where $\mathbf{E}_\lambda$ can be written as

$$\mathbf{E}_\lambda(x, x_0) = -\frac{1}{\lambda \int_{\mathbb{R}} \mathbf{q}(z)\mathbf{u}_*'(z)\mathrm{d}z}\mathbf{u}_*'(x)\mathbf{q}(x_0),$$

while the remainder term depends analytically of λ in the region $|\lambda| < \epsilon$.

We need estimates on $\widetilde{\mathbf{G}}_\lambda$ and we expect that

$$\left|\widetilde{\mathbf{G}}_\lambda(x, x_0)\right| \leq Ce^{-\omega'|x-x_0|}, \quad (\text{ongoing work})$$

for some $C > 0$ and $\omega' > 0$.

Related works: Beck etal '10, Hupkes-Sandstede '13, Schouten-Hupkes '17

Discussion

For Lattice Neural Field Equation

$$\dot{u}_n(t) = -u_n(t) + \sum_{j\in\mathbb{Z}} K_j S(u_{n-j}(t))$$

- existence & uniqueness (up to translation) of monotone traveling front solutions for "bistable" type of kinetics
- spectral stability of nonzero wave speed traveling fronts (need an exponential localization of interactions)
- toward a nonlinear stability result – study of pointwise Green's functions
- difficulty & novelty: infinite nonlinear range interactions
- study numerical approximation schemes of continuous NFE
- study other type of networks and/or kinetics (monostable, with linear adaptation)

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Research courses

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