Traveling fronts for lattice neural field equations

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Excitatory neural network on a lattice

Lattice Neural Field Equation (LNFE)

\[ \dot{u}_n(t) = -u_n(t) + \sum_{j \in \mathbb{Z}} K_j S(u_{n-j}(t)), \quad (n, t) \in \mathbb{Z} \times (0, \infty) \]

- \( u_n(t) \in \mathbb{R} \): membrane potential of neuron labelled \( n \) at time \( t \);
- \( K_j \geq 0 \): strength of interactions (all to all – infinite range)
- \( u \mapsto S(u) \): firing rate function (of sigmoidal type)
- discrete version of

\[ \partial_t u(x, t) = -u(x, t) + \int_{\mathbb{R}} K(y) S(u(x - y, t)) \, dy \]

References: Wilson-Cowan ’72, Amari ’77, Hopfield’84, Ermentrout ’98, Coombes ’05, Bressloff ’12
Traveling waves

Traveling wave: speed $c \in \mathbb{R}$

$$u_n(t) = u(n - ct), \quad u : \mathbb{R} \rightarrow \mathbb{R}$$

Traveling wave equation: $x = n - ct$

$$-cu'(x) = -u(x) + \sum_{j \in \mathbb{Z}} K_j S(u(x - j)), \quad x \in \mathbb{R}, \quad (1a)$$

$$\lim_{x \to -\infty} u(x) = 1 \text{ and } \lim_{x \to +\infty} u(x) = 0, \quad (1b)$$

- FDE of mixed type (ill posed as a Cauchy problem)
- dynamics depend on (all) past $j \leq 0$ and (all) future $0 \leq j$

It is assumed that when $c = 0$ equation (1) is an infinite recurrence.
Assumptions

On the firing rate function $S$:

**Hypothesis (H1) - Bistable nonlinearity.**  We suppose that:

(i) $S \in C^r_b(\mathbb{R})$ for $r \geq 2$ with $S(0) = 0$ and $S(1) = 1$ together with $S'(0) < 1$ and $S'(1) < 1$;

(ii) there exists a unique $\theta \in (0, 1)$ such that $S(\theta) = \theta$ with $S'(\theta) > 1$;

(iii) $u \mapsto S(u)$ is strictly nondecreasing on $[0, 1]$ and there exists $s_m > 1 > s_0 > 0$ such that $s_0 < S'(u) \leq s_m$ for all $u \in [0, 1]$.

On the weights $(K_j)_{j \in \mathbb{Z}}$:

**Hypothesis (H2) - Weights.**  We suppose that:

(i) $\sum_{n \in \mathbb{Z}} K_n = 1$;

(ii) for all $n \in \mathbb{Z}$, we have $K_n = K_{-n} \geq 0$ and $K_{\pm 1} > 0$;

(iii) $\sum_{n \in \mathbb{Z}} |n| K_n < \infty$. 
## Existence of monotone traveling waves

**Theorem**

*Suppose that the Hypotheses (H1)-(H2) are satisfied then there exists a traveling wave solution* \( u_n(t) = u_*(n - c_\ast t) \) *of LNFE such that the profile* \( u_\ast \) *satisfies the traveling wave problem (1).*

*Moreover:*

1. \( \text{sgn}(c_\ast) = \text{sgn} \int_0^1 (-u + S(u)) du \) if \( c_\ast \neq 0 \);
2. if \( \int_0^1 (-u + S(u)) du = 0 \) then \( c_\ast = 0 \);
3. if \( c_\ast \neq 0 \) then \( u_\ast \in C^{r+1}(\mathbb{R}) \) and \( u'_\ast < 0 \) on \( \mathbb{R} \);
4. if \( c_\ast = 0 \) we denote \( (\tilde{u}_n^\ast)_{n \in \mathbb{Z}} \) the stationary wave solution, then \( (\tilde{u}_n^\ast)_{n \in \mathbb{Z}} \) is a strictly decreasing sequence.
Sketch of the proof

The key idea (Bates & Chen '99) is to regularize the traveling wave equation

\[-cu'(x) = -u(x) + \sum_{j \in \mathbb{Z}} K_j S(u(x - j))\]

Let \(\Psi \in C^\infty(\mathbb{R}), \Psi \geq 0, \int_{\mathbb{R}} \Psi(x)dx = 1\), even and with compact support

\[\rho_m(x) := m\Psi(mx), \quad \mathcal{K}_m(x) := \sum_{j=-m}^{m} \frac{1}{\omega_m} K_j \rho_m(x - j), \text{ with } \omega_m := \sum_{j=-m}^{m} K_j\]

New traveling wave problem

\[-c_m u'_m = -u_m + \mathcal{K}_m \ast S(u_m), \text{ on } \mathbb{R}, \quad (2a)\]
\[\lim_{x \to -\infty} u_m(x) = 1 \text{ and } \lim_{x \to +\infty} u_m(x) = 0. \quad (2b)\]

Apply the results of Ermentrout & McLeod '93 to get a monotone solution \((u_m, c_m)\) of (2) then pass to the limit \(m \to +\infty\).
Uniqueness of traveling waves with nonzero speed

Theorem

Let \((u_*, c_*)\) be a solution to (1) as given in Theorem 1, such that \(c_* \neq 0\). Let \((\hat{u}, \hat{c})\) be another solution to (1). Then \(c = \hat{c}\) and, up to a translation, \(u_* = \hat{u}\).

Ideas of the proof:

- Construct appropriate sub and super solutions of the form

\[w_n^\pm(t) := u_* \left( n - c_* t + \xi_0 \mp \sigma \gamma (1 - e^{-\beta t}) \right) \pm \gamma e^{-\beta t}, \quad \forall n \in \mathbb{Z}\]

for some parameters \(\xi_0, \sigma, \gamma\) and \(2\beta = \min \{1 - S'(0); 1 - S'(1)\}\).

- Use comparison principle and a "squeezing" technique to prove uniqueness

Reference: Chen'97
Spectral stability

LNFE in moving coordinate:

\[-cu'(x) = -u(x) + \sum_{j \in \mathbb{Z}} K_j S(u(x - j))\]

Linearized operator around \((u_*, c_*)\) with \(c_* \neq 0\):

\[\mathcal{L} : H^1(\mathbb{R}) \to L^2(\mathbb{R}), \quad \mathcal{L}v := c_* v' - v + K_\delta * [S'(u_*) v],\]

where \(K_\delta * w = \sum_{j \in \mathbb{Z}} K_j w(\cdot - j)\).

Floquet-like spectral structure:

\[\mathcal{L}(e^{2\pi ix} u) = e^{2\pi ix} (2\pi ic_* + \mathcal{L})u\]

- spectrum invariant under shifts by \(2\pi ic_*\)
- lattice doesn’t feel oscillations on scale smaller than distance in lattice
Spectral properties of $\mathcal{L}$

**Hypothesis (H2$\eta$) - Exponential localization.** We suppose that:

(i) $(K_j)_{j \in \mathbb{Z}}$ satisfies (H2);

(ii) there exists $\eta > 0$, such that $\sum_{j \in \mathbb{Z}} K_j e^{\eta |j|} < \infty$.

**Proposition**

Assume that Hypotheses (H1)-(H2$\eta$) and that $(u_*, c_*)$ is the traveling wave solution given in Theorem 1 with $c_*$. We have:

- $0$ is an algebraically simple eigenvalue of $\mathcal{L}$ with a negative eigenfunction $u'_*$;

- the adjoint operator $\mathcal{L}^*$ has a negative eigenfunction, denoted $q \in C^1(\mathbb{R})$, corresponding to the simple eigenvalue $0$;

- for all $0 < \kappa < 2\beta$ the operator $\mathcal{L} - \lambda$ is invertible as an operator from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$ for all $\lambda \in \mathbb{C} \setminus 2\pi i c_* \mathbb{Z}$ such that $\Re(\lambda) \geq -\kappa$;

- there exist $\eta_* , \eta^{**} \in (0, \eta)$ and some constants $C_* > 0$, $C^{**} > 0$ such that

$$|u'_*(x)| \leq C_* e^{-\eta_* |x|} \|u_*\|_{L^\infty(\mathbb{R})}, \text{ and } |q(x)| \leq C^{**} e^{-\eta^{**} |x|} \|q\|_{L^\infty(\mathbb{R})}.$$
Toward nonlinear stability

**GOAL:** given existence and spectral stability, prove nonlinear stability

LNFE can be written

\[ \dot{u}(t) = F(u(t)), \text{ with } u = (u_n)_{n \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z} \]

where \( \bar{u}(t) = (u_*(n - c_* t))_{n \in \mathbb{Z}} \) with \( c_* \neq 0 \) is a solution with

\[ \bar{u}_n(t) = \bar{u}_{n-1} \left( t - \frac{1}{c_*} \right), \quad n \in \mathbb{Z} \]

- TW is relative periodic orbit
- Linearization:

\[ \dot{v}(t) = D\mathcal{F}(\bar{u}(t))v(t) \]

⇒ use spectral information on \( \mathcal{L} \) to obtain decay estimates on the flow of \( \dot{v}(t) = D\mathcal{F}(\bar{u}(t))v(t) \) and prove nonlinear stability
Main strategy

1. Moving coordinate frame:

\[ \partial_t \mathbf{v} = \mathbf{L} \mathbf{v}, \quad (\mathbf{L} - \lambda) \mathbf{G} = \delta(\cdot - x_0), \]

- spectral properties on \( \mathbf{L} \)
- resolvent kernel \( \mathbf{G}_\lambda(x, x_0) \)

2. Coordinates of the lattice:

\[ \mathbf{v}(t) = D\mathcal{F}(\mathbf{u}(t))\mathbf{v}(t), \quad \mathbf{v}(t_0) = (\delta(j - j_0))_{j \in \mathbb{Z}} \quad \Rightarrow \text{solution } \mathbf{v}^{t_0j_0}(t) \]

- Green’s function

\[ \mathcal{G}_{jj_0}(t, t_0) = \mathbf{v}_j^{t_0j_0}(t) \]

- Relationship (Benzoni-Gavage, Huot, Rousset ’03)

\[ \mathcal{G}_{jj_0}(t, t_0) = \frac{1}{2\pi i} \int_{R-i\pi c_*}^{R+i\pi c_*} e^{\lambda(t-t_0)} \mathbf{G}_\lambda(j-c_*t, j_0-c_*t_0) d\lambda \quad R \gg 1 \]
Green's function representation

**Proposition**

Assume that Hypotheses (H1)-(H2) and that \((u_*, c_*)\) is the traveling wave solution given in Theorem 1 with \(c_*\). Then there exists \(\epsilon > 0\) such that for all \(\lambda \in \mathbb{C}\) with \(0 < |\lambda| < \epsilon\), we have the representation

\[
G_\lambda(x, x_0) = E_\lambda(x, x_0) + \tilde{G}_\lambda(x, x_0),
\]

where \(E_\lambda\) can be written as

\[
E_\lambda(x, x_0) = -\frac{1}{\lambda \int_{\mathbb{R}} q(z)u_*(z)dz} u'(x)q(x_0),
\]

while the remainder term depends analytically of \(\lambda\) in the region \(|\lambda| < \epsilon\).

We need estimates on \(\tilde{G}_\lambda\) and we expect that

\[
|\tilde{G}_\lambda(x, x_0)| \leq Ce^{-\omega'|x-x_0|}, \quad \text{(ongoing work)}
\]

for some \(C > 0\) and \(\omega' > 0\).

**Related works:** Beck et al '10, Hupkes-Sandstede '13, Schouten-Hupkes '17
Discussion

For Lattice Neural Field Equation

\[ \dot{u}_n(t) = -u_n(t) + \sum_{j \in \mathbb{Z}} K_j S(u_{n-j}(t)) \]

- existence & uniqueness (up to translation) of monotone traveling front solutions for "bistable" type of kinetics
- spectral stability of nonzero wave speed traveling fronts (need an exponential localization of interactions)
- toward a nonlinear stability result – study of pointwise Green’s functions
- difficulty & novelty: infinite nonlinear range interactions
- study numerical approximation schemes of continuous NFE
- study other type of networks and/or kinetics (monostable, with linear adaptation)
Thematic semester on Mathematics Computer science and biology

Deterministic and Stochastic Models in Neurosciences

WINTER SCHOOL

Research courses
- François Delarue
- Eva Löcherbach
- Delphine Salort
- Romain Veltz

Invited speakers
- Daniele Avitabile
- Bruno Cessac
- Zachary Kilpatrick
- Carlo Laing
- Cristobal Quininao
- Wilhelm Stannat

Organizers
Patrick Cattiaux & Grégory Faye

http://www.cimi.univ-toulouse.fr/mib/

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