Renewal in Hawkes processes with self-excitation and inhibition

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Point processes

★ We consider point measures on $\mathbb{R}$. $\mathcal{N}(\mathbb{R})$ is the set of counting measures embedded with the topology of vague convergence.
For a point measure N, a Borel set E and a measurable function f:

$$N(f) = \int_{\mathbb{R}} f(x)N(dx) = \sum_{x \in \operatorname{supp} N} f(x),$$

$$N(E) = N(\mathbf{1}_E) = \operatorname{Card}(\operatorname{supp}(N) \cap E).$$

★ Given a filtration $(\mathcal{F}_t)_{t\geq 0}$, the conditional intensity is the function Λ s.t.

$$\Lambda(t) = \lim_{h \to 0} \frac{1}{h} \mathbb{E}\big(N([t, t+h)) \mid \mathcal{F}_t\big).$$

We have:

$$\mathbb{E}\big(N(f)\big) = \mathbb{E}\Big(\int_{\mathbb{R}} f(s)\Lambda(s)ds\Big).$$

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★ An example of random point measure is the Poisson point process (PPP) with intensity $\Lambda \equiv \lambda$.
The PPP exhibits a 'lack of memory' property.

Definition of Hawkes processes

★ **Def:** Let $\lambda > 0$, h be a measurable function from $(0, +\infty) \to \mathbb{R}$ and $\mathfrak{m}$ be a probability measure on $\mathcal{N}((-\infty, 0))$.

The point process N^h on $\mathbb{R}$ is a Hawkes process on $(0, +\infty)$ with initial condition $N^0 \rightsquigarrow \mathfrak{m}$ if the conditional intensity measure of $N^h|_{(0,+\infty)}$ is absolutely continuous w.r.t. the Lebesgue measure with density:

$$\Lambda^h(t) = \left(\lambda + \int_0^t h(t-u)N^h(du)\right)^+ = \left(\lambda + \sum_{u \le t, u \in N^h} h(t-u)\right)^+.$$

★ The function h is the reproduction function.

$h > 0$: self-excitation.
$h < 0$: inhibition.

The function h encodes for an underlying age-structure.

Motivations

★ Modelling of earthquakes (Hawkes Adamopoulos 73, Ogata 88),

★ financial markets (Bacry et al. 2011)

★ Modelling neuron transmissions (Reynaud 13, Delattre Fournier Hoffmann 16, Chevallier 17, Hodara Löcherbach 17...)

In particular, in the mean-field limit of large networks of neurons (Delattre et al., Chevallier 17, Ditlevsen and Löcherbach 17) we have propagation of chaos characterized by 'typical' nonlinear PP, related with ODEs and PDEs.

Aims:

★ Existence, Uniqueness, Coupling with h^+

★ Probability toolbox, for statistics ?

For example Reynaud-Bouret and Roy establish deviation inequalities in the case $h \geq 0$ for

$$\frac{1}{T}\int_0^T f\big(N^h(.+t)\big)dt.$$

Assumptions on h

★ $L(h) = \sup\{t > 0,\ |h(t)| > 0\} < +\infty$.

★ $\|h^+\|_1 < 1$.

★ $\mathbb{E}_{\mathfrak{m}}\left(N^0(-L(h), 0]\right) < +\infty$.

★ **Prop:** Under these assumptions, there is no explosion of the Hawkes processes (assuming existence).

$$\mathbb{E}_{\mathfrak{m}}\left(N^h(0, t \wedge U_k^h)\right) = \mathbb{E}_{\mathfrak{m}}\left(\int_0^{t \wedge U_k^h} \Lambda^h(u)du\right)$$

$$= \mathbb{E}_{\mathfrak{m}}\left(\int_0^{t \wedge U_k^h} \left(\lambda + \int_{(-\infty,u)} h(u-s)N^h(ds)\right)^+ du\right)$$

$$\leq \lambda t + \mathbb{E}_{\mathfrak{m}}\left(\int_0^{t \wedge U_k^h} \int_{(-\infty,0]} h^+(u-s)N^0(ds)du\right) + \mathbb{E}_{\mathfrak{m}}\left(\int_0^{t \wedge U_k^h} \int_{(0,u)} h^+(u-s)N^h(ds)du\right)$$

$$\leq \lambda t + \|h^+\|_1 \mathbb{E}_{\mathfrak{m}}\left(N^0(-L(h), 0]\right) + \|h^+\|_1 \mathbb{E}_{\mathfrak{m}}\left(N^h(0, t \wedge U_k^h)\right)$$

implying

$$0 \leq \mathbb{E}_{\mathfrak{m}}\left(N^h(0, t \wedge U_k^h)\right) \leq \frac{1}{1 - \|h^+\|_1}\left(\lambda t + \|h^+\|_1 \mathbb{E}_{\mathfrak{m}}\left(N^0(-L(h), 0]\right)\right)$$

Existence and uniqueness (1)

★ Equation driven by a Poisson point measure.

$$N^h = N^0 + \int_{(0,+\infty)^2} \delta_u \mathbf{1}_{\theta \le \Lambda^h(u)} Q(du, d\theta),$$

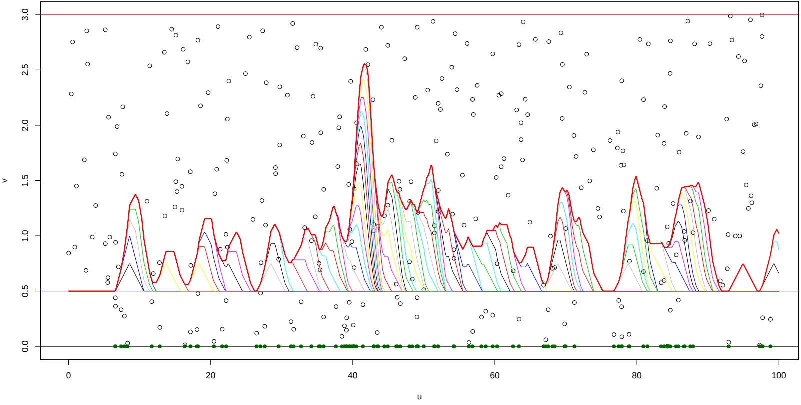
$$\forall u > 0,\ \Lambda^h(u) = \left(\lambda + \int_{(-\infty,u)} h(u-s) N^h(ds)\right)^+ .$$

★ Picard iteration method: see Brémaud Massoulié (96).

★ In the case where $h \geq 0$, we have a cluster representation of N^h:

(i) 'Ancestors' immigrate with rate λ,
(ii) An atom at time s is considered as an individual born at time s, with lifetime $L(h)$.
It gives birth with rate $h(. - s)$:
The offspring number is a Poisson random variable with parameter $\|h\|_1$.
The birth times of these offspring are drawn independently in $(s, s + L(h))$ with density $h(.)/\|h\|_1$.

Existence and uniqueness (2)



★ The Hawkes process is then the superposition of these sub-critical age-structured Galton-Watson trees.

★ For general h, the cluster representation is no more true. Pruned tree representation (complicated).

Existence and uniqueness (3)

★ Equation driven by a Poisson point measure.

$$N^h = N^0 + \int_{(0,+\infty)^2} \delta_u \mathbf{1}_{\theta \leq \Lambda^h(u)} Q(du, d\theta),$$

$$\forall u > 0,\ \Lambda^h(u) = \left(\lambda + \int_{(-\infty,u)} h(u-s) N^h(ds)\right)^+ .$$

★ Using coupling methods, we can show that:

Prop: If $\|h^+\|_1 < 1$ and $\mathbb{E}_{\mathfrak{m}}(N^0(-A,0]) < +\infty$,

(i) There exists a unique strong solution of the Equation driven by Q.
(ii) For all Borel set B of $\mathbb{R}$, $N^h(B) \leq N^{h^+}(B)$ a.s.

★ This allows to control N^h with N^{h^+}.

Auxiliary Markov process

★ Let $A > L(h)$ be a window of interest.

★ Assume that we are interested in the asymptotic properties of

$$\frac{1}{T}\int_0^T f\big(N^h(. + t)\big)dt,$$

where f is a function on $\mathcal{N}(-A, 0]$ that is locally bounded w.r.t. the variation norm.

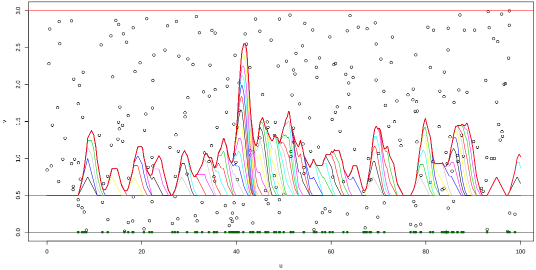
★ It is natural to introduce:

$$X_t = (S_t N^h)|_{(-A,0]} = N^h|_{(t-A,t]}(. + t).$$

★ **Prop:** The process $(X_t)_{t\geq 0}$ is a strong Markov process with initial condition $X_0 = N^0|_{(-A,0]}$ in $\mathbb{D}(\mathbb{R}_+, \mathcal{N}(-A, 0])$.

For Markov processes, the properties of $\frac{1}{T}\int_0^T f(X_t)dt$ can be studied by renewal properties.

Renewal times



★ We define for N^h (or X):

$$\begin{aligned}\tau &= \inf\{t > 0 : X_{t-} \neq \emptyset, X_t = \emptyset\} \\ &= \inf\{t > 0 : N^h[t - A, t) \neq 0, N^h(t - A, t] = 0\}\end{aligned}$$

Ergodic theorem

★ **Th:**

$$\frac{1}{T}\int_0^T f((S_t N^h)|_{(-A,0]})\,dt \xrightarrow[T\to\infty]{\mathbb{P}_{\mathfrak{m}}-\text{a.s.}} \pi_A f = \frac{1}{\mathbb{E}_\emptyset(\tau)}\mathbb{E}_\emptyset\left(\int_0^\tau f(X_t)\,dt\right) = \frac{1}{\mathbb{E}_\emptyset(\tau)}\mathbb{E}_\emptyset\left(\int_0^\tau f(N|_{(t-A,t]}(\cdot+t))\,dt\right).$$

Moreover,

$$\mathbb{P}_{\mathfrak{m}}\left((S_t N^h)|_{[0,\infty)} \in \cdot\right) \xrightarrow[t\to\infty]{\text{total variation}} \mathbb{P}_{\pi_A}(N^h \in \cdot)$$

Idea of proof

★ The idea is to decompose the path of (X_t) into independent excursions outside $\emptyset$.

$$\int_0^T f(X_t)\,dt = \int_0^{\tau_0} f(X_t)\,dt + \sum_{k=1}^{K_T} I_k f + \int_{\tau_{K_T}}^T f(X_t)\,dt$$

where $I_k f = \int_{\tau_{k-1}}^{\tau_k} f(x_t)dt$ and $K_T = \max\{k \geq 0,\ \tau_k \leq T\}$.

★ The strong law of large numbers yields that

$$\frac{1}{K_T}\sum_{k=1}^{K_T+1} I_k f \xrightarrow[T\to\infty]{\mathbb{P}_{\mathfrak{m}}-\text{a.s.}} \mathbb{E}(I_1 f) = \mathbb{E}_{\emptyset}\Big(\int_0^{\tau} f(X_t)\,dt\Big).$$

Central Limit theorem

★ **Th:** As soon as

$$\sigma^2(f) = \frac{1}{\mathbb{E}_\emptyset(\tau)} \mathbb{E}_\emptyset\left(\left(\int_0^\tau (f(N^h|_{(t-A,t]}(\cdot + t)) - \pi_A f)\, dt \right)^2 \right) < \infty$$

then:

$$\sqrt{T}\left(\frac{1}{T} \int_0^T f(N^h|_{(t-A,t]}(\cdot + t))\, dt - \pi_A f \right) \xrightarrow[T\to\infty]{\text{in law}} \mathcal{N}(0, \sigma^2(f)) \,.$$

Deviation inequalities

★ **Prop:** Let $\varepsilon > 0$. Assume also that $a \leq f \leq b$. There exists constants $C(f)$ and $D(f)$ such that:

$$\mathbb{P}_\emptyset\Big(\Big|\frac{1}{T}\int_0^T f(X_t)dt - \pi_A f\Big| \geq \frac{1}{T}\Big((b-a)\mathbb{E}_\emptyset(\tau) + \frac{C(f)}{2}\log\big(\frac{4}{\varepsilon}\big) + \frac{1}{2}\sqrt{C^2(f)\log^2\big(\frac{4}{\varepsilon}\big) + 4TD(f)\log\big(\frac{4}{\varepsilon}\big)}\Big)\Big) \leq \varepsilon.$$

Sketch of proof

★ We have to establish deviation inequalities for τ. It is natural to introduce the same quantity τ^+ for N^{h^+}.

$\mathbb{P}_{\mathfrak{m}}(\tau \leq \tau^+) = 1$.

★ It now remains to study τ^+, i.e. the case when $h \geq 0$.

We have the cluster representation, and the problem is associated with a problem of queues.

Queueing processes

★ Assume $h \geq 0$. Let us denote by H the height of one of these G.W. trees.

$$\mathbb{P}(H > x) \leq C \exp(-\gamma x),$$

where $\gamma = (\|h\|_1 - \log(\|h\|_1) - 1)/L(h)$ and $C = \exp(1 - \|h\|_1)$.

★ $M/G/\infty$ queue: arrivals at rate λ, service duration $H + A$, infinite number of servers.

★ **Prop:** Assume that $h \geq 0$.

(i) If $\lambda < \gamma$, then $\mathbb{P}_\emptyset(\tau > x) = O(e^{-\lambda x})$,

(ii) If $\gamma \leq \lambda$, then for any $\alpha < \gamma$, $\mathbb{P}_\emptyset(\tau > x) = O(e^{-\alpha x})$.

(proof based on a formula by Takacs for the Laplace transform $\mathbb{E}(e^{-sB})$ of the busy period, then work for showing that the abscissa of convergence is $\sigma_c \leq -\gamma$).